

**A GLIMPSE ON FINDING INTEGER SOLUTIONS TO
HOMOGENEOUS QUATERNARY QUADRATIC DIOPHANTINE
EQUATION**

$$x y - 2 z w + (z - w)^2 = (x + y) (z + w)$$

N. Thiruniraiselvi^{*1}, M. A. Gopalan²

¹ Associate Professor, Department of Mathematics, Sri Ramakrishna College of Engineering, Affiliated to Anna University, Chennai, Sri Saradha Nagar, Perambalur-621 113, Tamil Nadu, India.

² Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Tiruchirappalli-620002, Tamil Nadu, India.

Article Received: 15 April 2026, Article Revised: 05 May 2026, Published on: 25 May 2026

***Corresponding Author: N. Thiruniraiselvi**

Associate Professor, Department of Mathematics, Sri Ramakrishna College of Engineering, Affiliated to Anna University, Chennai, Sri Saradha Nagar, Perambalur-621 113, Tamil Nadu, India.

DOI: <https://doi-doi.org/101555/ijarp.7676>

ABSTRACT

This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $x y - 2 z w + (z - w)^2 = (x + y) (z + w)$. Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable ternary quadratic equation. A few relations among the solutions are presented.

KEYWORDS: Quaternary quadratic equation, Homogeneous quadratic equation, Integer solutions.

Notations

$$t_{3,n} = \frac{n(n+1)}{2}$$

$$P_n^3 = \frac{n(n+1)(n+2)}{6}$$

$$P_n^4 = \frac{n(n+1)(2n+1)}{6}$$

$$P_n^5 = \frac{n^2(n+1)}{2}$$

INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety. In particular, quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [1-17] for second degree diophantine equations with three and two unknowns representing different geometrical figures. This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy - 2zw + (z - w)^2 = (x + y)(z + w)$. Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable ternary quadratic equation. A few relations among the solutions are presented.

Method of analysis

The polynomial equation of second degree with four unknowns to be solved is

$$xy - 2zw + (z - w)^2 = (x + y)(z + w) \quad (1)$$

The option

$$x = u + p, y = u - p, z = u + q, w = u - q, u \neq \pm p, \pm q \quad (2)$$

in (1) gives

$$p^2 + 5u^2 = 6q^2 \quad (3)$$

The procedure to obtain patterns of integer solutions to (1) is as below:

Pattern 1

Assume

$$q = a^2 + 5b^2 \quad (4)$$

Write the integer 6 as

$$6 = (1+i\sqrt{5})(1-i\sqrt{5}) \quad (5)$$

Substitute (4) & (5) in (3). Employing the method of factorization and equating the positive factors, consider

$$p+i\sqrt{5}u = (1+i\sqrt{5})(a+i\sqrt{5}b)^2$$

Equating the coefficients of corresponding terms, we have

$$p = a^2 - 5b^2 - 10ab$$

$$u = a^2 - 5b^2 + 2ab$$

In view of (2), the integer solutions to (1) are given by

$$x = 2a^2 - 10b^2 - 8ab$$

$$y = 12ab$$

$$z = 2a^2 + 2ab$$

$$w = -10b^2 + 2ab$$

(6)

A few numerical solutions to (1) are presented in Table 1 below:

Table 1- Numerical solutions.

a	b	x	y	z	w
1	1	-16	12	4	-8
2	3	-130	72	20	-78
3	4	-238	144	42	-136
6	2	-64	144	96	-16
7	5	-432	420	168	-180

Observations

1. $2(x - w) + 25$ is a perfect square
2. Each of the following expressions is expressed as the difference of two squares:
 $\frac{2}{5}(z - x), 2z - y$
3. $12z - 2y$ is a square multiple of 6
4. $12(x - w) + 10y$ is a nasty number
5. The triple $(x, z + w, 4x + 2y)$ is such that the product of any two members added with $\frac{y^2}{4}$ is a perfect square
6. $z + w - x$ is a nasty number when $a=2b$
7. $z + w - x = 24t_{3,b}$ when $a=b+1$

8. $z + w - x = 24P_b^5$ when $a=b (b+1)$
9. $z + w - x = 72P_b^3$ when $a=(b+1) (b+2)$
10. $z + w - x = 72P_b^4$ when $a=(b+1) (2 b+1)$
11. $3x + 2y$ is a square multiple of 6 when $a = 5p^2 + q^2, b = 2pq$

Note 1

In addition to (5) , the integer 6 may be expressed as

$$6 = \frac{(7 + i\sqrt{5})(7 - i\sqrt{5})}{9}$$

$$6 = \frac{(17 + i\sqrt{5})(17 - i\sqrt{5})}{49}$$

Following the above procedure , two more patterns of integer solutions to (1) are obtained.

Pattern 2

Write (3) as

$$p^2 + 5u^2 = 6q^2 * 1 \quad (7)$$

Assume the integer 1 on the R.H.S. of (7) as

$$1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{9} \quad (8)$$

Substitute (4) ,(5) & (8) in (7). Utilizing factorization and equating positive factors , we have

$$p + i\sqrt{5}u = (1 + i\sqrt{5})(a + i\sqrt{5}b)^2 \frac{(2 + i\sqrt{5})}{3}$$

$$= (-1 + i\sqrt{5})(a + i\sqrt{5}b)^2$$

On equating the real and imaginary parts in the above equation, one has

$$p = -(a^2 - 5b^2) - 10ab$$

$$u = (a^2 - 5b^2) - 2ab$$

From (2) , the corresponding integer solutions to (1) are given by

$$x = -12ab$$

$$y = 2a^2 - 10b^2 + 8ab$$

$$z = 2a^2 - 2ab \quad (9)$$

$$w = -10b^2 - 2ab$$

Note 2

Apart from (8) , the integer 1 may be taken as below:

$$1 = \frac{(5r^2 - s^2 + i\sqrt{5}(2rs))(5r^2 - s^2 - i\sqrt{5}(2rs))}{(5r^2 + s^2)^2}$$

$$1 = \frac{(10s^2 - 10s + 2 + i\sqrt{5}(2s-1))(10s^2 - 10s + 2 - i\sqrt{5}(2s-1))}{(10s^2 - 10s + 3)^2}$$

$$1 = \frac{(2s^2 - 2s - 2 + i\sqrt{5}(2s-1))(2s^2 - 2s - 2 - i\sqrt{5}(2s-1))}{(2s^2 - 2s + 3)^2}$$

$$1 = \frac{(5s^2 - 1 + i\sqrt{5}(2s))(5s^2 - 1 - i\sqrt{5}(2s))}{(5s^2 + 1)^2}$$

The repetition of the above process gives four more sets of integer solutions to (1).

Pattern 3

Consider (3) as

$$6q^2 - p^2 = 5u^2 \quad (10)$$

Assume

$$u = 6a^2 - b^2 \quad (11)$$

The integer 5 on the R.H.S. of (10) is written as

$$5 = (\sqrt{6} + 1)(\sqrt{6} - 1) \quad (12)$$

Substitute (11) & (12) in (10). Applying the method of factorization and equating the positive terms, one has

$$\sqrt{6}q + p = (\sqrt{6} + 1)(\sqrt{6}a + b)^2$$

from which we have

$$q = 6a^2 + b^2 + 2ab$$

$$p = 6a^2 + b^2 + 12ab$$

In view of (2), the integer solutions to (1) are given by

$$x = 12a^2 + 12ab$$

$$y = -2b^2 - 12ab \quad (13)$$

$$z = 12a^2 + 2ab$$

$$w = -2b^2 - 2ab$$

Note 3

Apart from (12), the integer 5 may be written as

$$5 = (3\sqrt{6} + 7)(3\sqrt{6} - 7)$$

$$5 = (7\sqrt{6} + 17)(7\sqrt{6} - 17)$$

Repeating the above process, one obtains two more patterns of integer solutions to (1).

Pattern 4

Write (3) as

$$p^2 = 6q^2 - 5u^2 \quad (14)$$

Inserting the transformations

$$q = X + 5T, u = X + 6T \quad (15)$$

in (14), we have

$$X^2 = p^2 + 30T^2 \quad (16)$$

which is satisfied by

$$T = 2rs, p = 30r^2 - s^2, X = 30r^2 + s^2 \quad (17)$$

Using (17) in (15), we get

$$q = 30r^2 + s^2 + 10rs$$

$$u = 30r^2 + s^2 + 12rs$$

In view of (2), the integer solutions to (1) are given by

$$x = 60r^2 + 12rs$$

$$y = 2s^2 + 12rs \quad (18)$$

$$z = 60r^2 + 2s^2 + 22rs$$

$$w = 2rs$$

Also, express (16) as the system of double equations as in Table 2 below:

Table 2—System of double equations.

System	I	II	III	IV	V	VI	VII	VIII
$X + p$	T^2	$3T^2$	$5T^2$	$15T^2$	$30T$	$15T$	$10T$	$6T$
$X - p$	30	10	6	2	T	2T	3T	5T

Solving each of the above system of equations, the values of X, T, p are obtained. From (15), the values of q and u are determined. From (2), the corresponding integer solutions to (1) are found. For simplicity and brevity, the integer solutions to (1) from solving each of the above system of equations are presented:

Solutions from System I

$$x = 4s^2 + 12s, y = 30 + 12s, z = 4s^2 + 30 + 22s, w = 2s$$

Solutions from System II

$$x = 12s^2 + 12s, y = 10 + 12s, z = 12s^2 + 10 + 22s, w = 2s$$

Solutions from System III

$$x = 20s^2 + 12s, y = 6 + 12s, z = 20s^2 + 6 + 22s, w = 2s$$

Solutions from System IV

$$x = 60s^2 + 12s, y = 2 + 12s, z = 60s^2 + 2 + 22s, w = 2s$$

Solutions from System V

$$x = 72s, y = 14s, z = 84s, w = 2s$$

Solutions from System VI

$$x = 42s, y = 16s, z = 56s, w = 2s$$

Solutions from System VII

$$x = 32s, y = 18s, z = 48s, w = 2s$$

Solutions from System VII

$$x = 24s, y = 22s, z = 44s, w = 2s$$

CONCLUSION

Plenty of solutions in integers are presented in this paper for the quaternary homogeneous quadratic equation given by $xy - 2zw + (z - w)^2 = (x + y)(z + w)$ through employing substitution technique and factorization method. As quadratic equations (homogeneous or non-homogeneous) are plenty, one may search for patterns of integer solutions to other choices of multivariable quadratic equations.

REFERENCES

1. M.A.Gopalan, S.Vidhyalakshmi and T.R.Usharani, Integral Points on the Non-homogeneous Cone $2z^2 + 4xy + 8x - 4z + 2 = 0$, Global Journal of Mathematics and Mathematical Sciences, Vol. 2(1), Pp. 61-67, 2012.
2. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Observations on the ternary quadratic Diophantine equation $x^2 + 9y^2 = 50z^2$, International Journal of Applied Research, Vol.1 (2), Pp.51-53, 2015.
3. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Construction of Diophantine quadruple through the integer solution of ternary quadratic Diophantine equation $x^2 + y^2 = z^2 + 4n$, International Journal of Innovative Research in Engineering and Science, Vol.5(4), Pp.1-7, May-2015

4. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, Observations on the cone $z^2 = ax^2 + a(a-1)y^2$, International Journal of Multidisciplinary Research and Development, Vol.2 (9), Pp.304-305, Sep-2015.
5. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, A Study on Special Homogeneous Cone $z^2 = 24x^2 + y^2$, Vidyabharati International Interdisciplinary Research Journal, (Special Issue on “Recent Research Trends in Management, Science and Technology”-Part-3, pdf page- 330), Pg: 1203-1208, 2021.
6. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, On the Homogeneous Ternary Quadratic Diophantine Equation $6x^2 + 5y^2 = 34lz^2$, Vidyabharati International Interdisciplinary Research Journal, (Special Issue on “Recent Research Trends in Management, Science and Technology”-Part-4, pdf page- 318), Pg: 1612-1617, 2021.
7. M.A.Gopala, S.Vidhyalakshmi, J.Shanthi, V.Anbuvali, On Finding the integer solutions of Ternary Quadratic Diophantine Equation $3(x^2 + y^2) - 5xy = 36z^2$, International Journal of Precious Engineering Research and Application(IJPERA), volume 7, Issue 1, 34-38, may 2022
8. J.Shanthi, M. Parkavi “On Finding Integer Solutions To The Homogeneous Ternary Quadratic Diophantine Equation $2(x^2 + y^2) - 3xy = 32z^2$ ” International Journal of Research Publication and Reviews, Vol 4, no 1, pp 700-708, January 2023.
9. J.Shanthi, V. Anbuvali, M.A.Gopalan, S. Vidhyalakshmi, On finding integer solutions to the homogeneous cone $x^2 + (k^2 + 2k)y^2 = (k+1)^4 z^2$, International Journal of Mathematics and computer Research, Volume 11, Issue 07, Pp 3555-3557, July 2023.
10. M.A.Gopalan, S.Vidhyalakshmi and N.Thiruniraiselvi, A class of new solutions in integers to Ternary Quadratic Diophantine Equation $12(x^2 + y^2) - 23xy + 2x + 2y + 4 = 56z^2$, International Journal of Research Publication and Reviews, Vol 5 (5), Page – 3224-3226, May (2024).
11. J.Shanthi, M.A.Gopalan, A Glimpse on Homogeneous Ternary Quadratic Diophantine Equation $39(x^2 + y^2) + 72xy = 246z^2$, IARJSET, 11(9), 96-101, 2024
12. J.Shanthi, M.A.Gopalan, On Finding Integer Solutions to Quaternary Quadratic Diophantine Equation $x^2 - 6y^2 + 15z^2 = w^2$, IJONS, 15 (88), 89269-89276, 2025
13. N.Thiruniraiselvi, M.A.Gopalan, Technique on Solving a Binary Quadratic Diophantine

Equation $3x^2 + 5y^2 = 17^{2k+1}$, Indiana Journal of Multidisciplinary Research ,4(4),1-3,2024

14. N.Thiruniraiselvi,M.A.Gopalan,G.Selvarathi,On the Positive Pell Equation $y^2 = 14x^2 + 18$,International Journal of Advanced Multidisciplinary Research and Studies,2(3), 241-243,2022
15. M.A.Gopalan,J.Shanthi,N.Thiruniraiselvi,A Troupe of Special Second Degree Multivariable Polynomial Diophantine Equations with Integer Solutions,Deep Science Publishing,2025
16. N.Thiruniraiselvi,M.A.Gopalan,On the Positive Pell Equation $y^2 = 11x^2 + 22$, Academic Journal of Applied Mathematical Sciences, 6(7), 85-92, 2020
17. N.Thiruniraiselvi ,M.A.Gopalan,On Finding Integer Solutions to Homogeneous Ternary Quadratic Diophantine Equation $x^2 + (2k + 1)y^2 = (k + 1)^2 z^2$,Rattanakosin Journal of Science and Technology,7(3),252-258,2025