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A GRAPH STRUCTURAL APPROACH TO MAXIMAL ANTICHAIN GRAPHS IN COMBINATORIAL OPTIMIZATION

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ABSTRACT

This paper presents a graph structural approach to the study of maximal antichain graphs and their potential applications in combinatorial optimization. The concept of maximal antichains, derived from partially ordered sets (posets), provides a rich framework for constructing graphs that exhibit unique connectivity, dominance and covering properties. The results reveal that maximal antichain graphs possess higher level of structural efficiency and minimal redundancy making them prominent for optimization processes that require balance between connectivity and independence. The paper further explore how the structural characteristics of these graphs such as degree distribution, chromatic number and clique structure can be utilized to model and solve optimization problems. Furthermore the paper provides valuable insight into the study of combinatorial behaviours of maximal antichains graphs.

KEYWORDS: graph theory, maximal antichain graphs, combinatorial optimization, posets.

INTRODUCTION

Graph theory is a cornerstone of modern mathematics, offering a versatile framework to model relationships between entities in diverse fields such as computer science, operations research, and network analysis. A graph $G = (V, E)$ consists of a vertex set V representing entities and an edge set E representing pairwise connections. Within graph theory, the concept of partially ordered sets (posets) provides a structured way to study orderings derived from graphs, such as vertex reachability or subgraph inclusion.

A poset is a set equipped with a binary relation that is reflexive, antisymmetric, and transitive. In this context, an *antichain* is a subset of elements where no two elements are comparable under the partial order, meaning neither element precedes the other. A *maximal antichain* is an antichain that cannot be extended by including additional elements without violating the incomparability condition. In graphs, maximal antichains manifest in various forms, such as independent sets in comparability graphs or sets of non-nested subgraphs in a poset of subgraphs ordered by inclusion.

The study of maximal antichains is deeply rooted in combinatorial mathematics, with key results like Dilworth's theorem. This theorem has profound implications for graph structures, enabling applications in optimization and scheduling. Maximal antichains have practical significance in areas such as task scheduling, where they identify sets of non-conflicting tasks that can be executed simultaneously, and network design, where they model independent nodes or resources.

In Nigeria, with its growing technological and industrial sectors, such concepts can address challenges like optimizing telecommunications networks or managing resource allocation in agriculture and logistics, aligning with the career aspirations of mathematics graduates seeking to apply theoretical knowledge practically.

Definition of Some Basic Terms

Definition 1.1: Graph A *graph* is a mathematical structure denoted by $G = (V, E)$, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges.

Definition 1.2: Partially Ordered Set (Poset) A *partially ordered set (poset)* is a set equipped with a binary relation that is reflexive, antisymmetric, and transitive.

Definition 1.3: Chain A *chain* is a subset of elements in which every pair of elements is comparable under the partial order.

Definition 1.4: Antichain An *antichain* is a subset of a poset in which no two elements are comparable under the partial order.

Definition 1.5: Maximal Antichain A *maximal antichain* is an antichain that cannot be extended by adding another element without violating incomparability.

Definition 1.6: Dilworth's Theorem In any finite poset, the size of the largest antichain is equal to the minimum number of chains needed to partition the poset.

Definition 1.7: Comparability Graph A *comparability graph* is a graph in which the vertices represent the elements of a poset, and edges connect pairs of comparable elements.

Definition 1.8: Independent Set An *independent set* is a set of vertices in a graph such that no two vertices in the set are adjacent.

Definition 1.9: Clique A *clique* is a subset of vertices in a graph such that every pair of vertices in the subset is adjacent.

Definition 1.10: Combinatorial Optimization *Combinatorial optimization* is a field of mathematics and computer science that seeks to find an optimal object (e.g., maximum independent set, minimum vertex cover, shortest path, etc.) in a finite but often exponentially large collection of feasible objects.

Methodology

In this chapter, the research considers the formation of maximal antichain graphs and their applications in combinatorial optimization.

It begins as follows: Let the set of real numbers R with the usual order relation be ($\leq$), and let the set of subsets of a given set S, the power set of S, be denoted by P(S) with the subset relation ($\subseteq$).

Bipartite Graph Representation

Let a bipartite graph be represented by U and V such that

$U = \{U_1, U_2, U_3\}$, $V = \{V_1, V_2, V_3, V_4\}$

and the set of edges is given by:

$E = \{(U_1, V_1), (U_1, V_2), (U_2, V_1), (U_3, V_2), (U_3, V_3)\}$.

The neighborhood of the vertices in U is defined as:

$N(U_1) = \{V_1, V_2\}$,

$N(U_2) = \{V_1\}$,

$N(U_3) = \{V_2, V_3\}$.

Hence, the maximal antichain is given by:

$N(U_3) \leq N(U_2) \leq N(U_1)$.

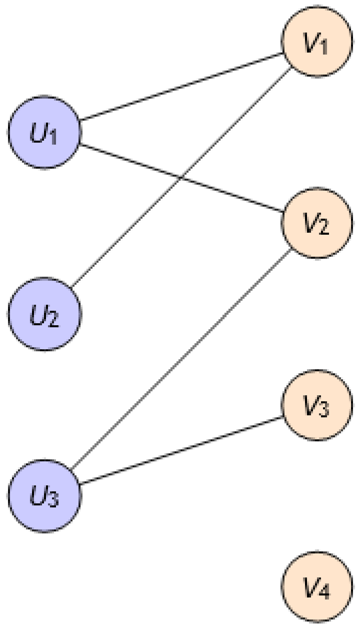


Figure 1: Bipartite graph representation showing neighborhood sets $N(U_1)$, $N(U_2)$, and $N(U_3)$.

1.1 Path Graph P_4

Let the vertices be $\{V_1, V_2, V_3, V_4\}$ and the edges be

$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_4)\}$.

The bipartition divides the vertices into two sets:

$U = \{V_1, V_3\}$, $V = \{V_2, V_4\}$.

The path graph P_4 is bipartite because the vertices can be alternated between two sets such that no edges connect vertices within the same set.

Example 3.3: Adjacency Matrix

$$A(P_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

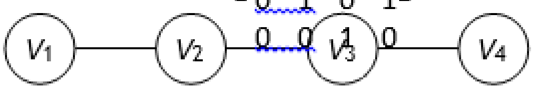


Figure 3.2: Path Graph P_4

1.2 Boolean Lattice

The power set of an n-element set ordered by inclusion forms a Boolean lattice. By Sperner's theorem, the largest antichain is the set of all subsets of size $\lfloor n/2 \rfloor$.

For $n = 4$, the subsets of size 2 form a maximal antichain:

A = $\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

Example 3.4: Power Set of a Three-Element Set Let $S = \{1, 2, 3\}$. Then $2^S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Operations:

$A \cap B$ (Meet), $A \cup B$ (Join), $S \setminus A$ (Complement).

Examples of Operations:

$\{1, 2\} \cap \{1, 3\} = \{1\}$, $\{1\} \cup \{2\} = \{1, 2\}$, $S \setminus \{1, 2\} = \{3\}$.

This forms a Boolean lattice with $2^3 = 8$ elements, represented as a three-dimensional cube structure.

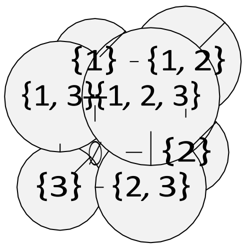


Figure 3.3: Boolean Lattice B_3

Boolean Lattice of Two Elements Let $S = \{a, b\}$. Then

$P(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Ordered by inclusion, the maximal antichain is $\{\{a\}, \{b\}\}$, since no two of these are comparable, and every other element is comparable to at least one of them.

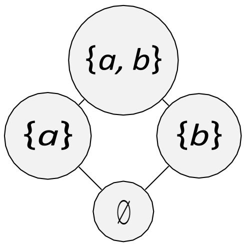
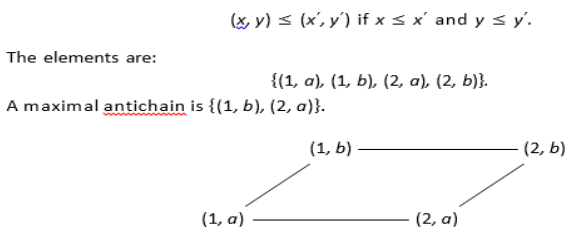


Figure 3.4: Boolean Lattice B_2

1.3 Products of Chains

Consider the poset formed by the product of two chains $\{1 < 2\}$ and $\{a < b\}$ with the component-wise order:



$(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

The elements are:

$\{(1, a), (1, b), (2, a), (2, b)\}$.

A maximal antichain is $\{(1, b), (2, a)\}$.

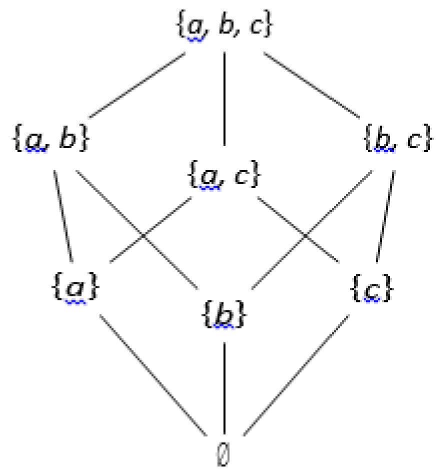
1.3 The Power Set of a 3-Element Set

Let $S = \{a, b, c\}$. The power set P(S) ordered by ⊆. One maximal antichain is the set of all 2-element subsets:

$$\{\{a, b\}, \{a, c\}, \{b, c\}\}.$$

Another is the set of all 1-element subsets:

$$\{\{a\}, \{b\}, \{c\}\}.$$



1.4 The Chain Poset C_3

A chain poset with three elements implies $\{a, b, c\}$ where $a < b < c$.

$$a < b < c$$

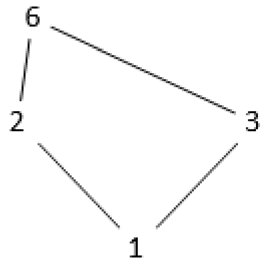
A maximal antichain example is $\{b\}$, which is a single node.

1.5 The Divisibility Poset on {1, 2, 3, 6}

Let us consider the set $\{1, 2, 3, 6\}$ with partial order defined by divisibility, i.e., $a \leq b$ if a divides b.

$a \leq b$ if a divides b.

The order relations are 1|2, 1|3, 1|6, 2|6, and 3|6.



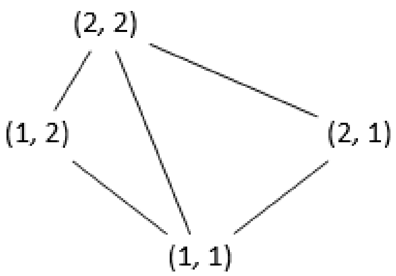
1.6 The Product Poset $\{1,2\}\times\{1,2\}$

The poset is the Cartesian product:

$$\{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

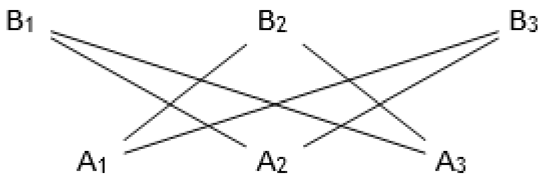
with component-wise order $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. Relations include:

$$(1, 1) \leq (1, 2), (2, 1), (2, 2); \quad (1, 2) \leq (2, 2); \quad (2, 1) \leq (2, 2).$$



1.7 The Crown Poset C_6

The crown poset on six elements, say $\{a_1, a_2, a_3, b_1, b_2, b_3\}$, where the order is defined such that $a_i < b_j$ for $i \neq j$. Incomparable pairs include all a_i's among themselves and all b_i's among themselves. The set $\{a_1, a_2, a_3\}$ or $\{b_1, b_2, b_3\}$ is a maximal antichain.



Theorem 2.1: Clem and West (2018)

In the Boolean lattice B_n of all subsets of $[n]$ ordered by inclusion, the largest antichain

(Sperner family) has size $\binom{n}{\lfloor n/2 \rfloor}$.

Proof: Each chain in B_n intersects each layer (subsets of size k) in at most one element. There are $n + 1$ layers (from size 0 to n). By Dilworth's Theorem, the width is the size of the largest layer, which is $\binom{n}{\lfloor n/2 \rfloor}$.

Theorem 2.2: Zhang et.al (2023)

In any finite poset P, the size of the largest chain is equal to the smallest number of antichains needed to cover all elements of P

Proof: This is the dual of Dilworth's Theorem. Assign levels to elements based on chain lengths, and each level forms an antichain. The number of levels equals the length of the largest chain.

3.0 RESULTS AND DISCUSSION

Proposition 3.1.

The width $W(P)$ of a poset P (the size of its largest antichain) is equal to the size of the largest set of mutually incomparable vertices in the associated comparability graph.

Proof: The comparability graph of P connects elements that are comparable in P. Hence, an antichain corresponds to an independent set in the graph. The largest antichain corresponds to the largest independent set, establishing $W(P)$ as the independence number of the comparability graph. □

Proposition 3.2.

In the Boolean lattice B_n of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion, the antichains are called *Sperner families*. The family of all subsets of size k forms an antichain, and the largest such antichain has size

$$\binom{n}{\lfloor n/2 \rfloor}.$$

Proof: Two subsets of $[n]$ of equal size are incomparable under inclusion. The largest layer in B_n consists of subsets of size $\lfloor n/2 \rfloor$, whose number is $\binom{n}{\lfloor n/2 \rfloor}$. Hence, this forms the largest antichain, also known as a Sperner family. □

Proposition 3.3.

In a bipartite poset $P = A \cup B$, if there exists a maximal antichain matching of size k, that is, a set of k pairwise incomparable pairs (a_i, b_i), then the interval dimension $\mathrm{Idim}(P) \leq k$.

Proof: Each incomparable pair (a_i, b_i) defines an interval order whose realization contributes to bounding the interval dimension. Since k such pairs suffice to cover all incomparabilities, the interval dimension of P is at most k. □

Lemma 3.1

Every finite poset P can be partitioned into layers of maximal antichains $A_1, A_2, \ldots, A_k$ such that if $x \in A_i$ and $y \in A_j$ with $i < j$, then $x < y$ in the poset.

Proof: Construct each layer by repeatedly removing maximal antichains from the poset. This guarantees comparability direction from lower to higher layers.

Theorem 3.1

In any finite poset derived from a graph G, the size of the largest antichain is equal to the minimum number of chains that partition the poset.

Proof: Let A be the largest antichain. Suppose there exists a chain partition with fewer chains than $|A|$, then some chains must contain two elements of A, which contradicts the definition of an antichain. No two elements in an antichain are comparable, and elements in a chain are comparable.

Thus, any chain partition must have at least k chains. Now we show that there exists a chain partition with exactly k chains. Hence, the minimum number of chains is equal to the size of the largest antichain.

4.0 CONCLUSION

This research advances the mathematical understanding of maximal antichains and their role in combinatorial optimization. By bridging theoretical graph theory with practical applications, the study offers valuable insights for mathematicians, contributing to the growing body of knowledge in discrete mathematics and optimization theory.

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