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A GLIMPSE ON FINDING INTEGER SOLUTIONS TO HOMOGENEOUS QUATERNARY QUADRATIC DIOPHANTINE EQUATION

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ABSTRACT

This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy-2zw+(z-w)^2=(x+y)(z+w)$ .Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduce the quadratic equation with four unknowns to solvable ternary quadratic equation. A few relations among the solutions are presented.

KEYWORDS: Quaternary quadratic equation, Homogeneous quadratic equation, Integer solutions.

Notations

$$xy-2zw+(z-w)^2=(x+y)(z+w)$$

$$t_{3,n}=\frac{n(n+1)}{2}$$

$$P_n^3=\frac{n(n+1)(n+2)}{6}$$

$$P_n^4=\frac{n(n+1)(2n+1)}{6}$$

$$P_n^5=\frac{n^2(n+1)}{2}$$

INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety. In particular, quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [1-17] for second degree Diophantine equations with three and two unknowns representing different geometrical figures. This paper aims at finding patterns of solutions in integers to quaternary quadratic diophantine equation given by $xy-2zw+(z-w)^2=(x+y)(z+w)$ .Substitution technique and factorization method are utilized to obtain varieties of integer solutions. It is worth to observe that the introduction of the transformations reduces the quadratic equation with four unknowns to solvable ternary quadratic equation. A few relations among the solutions are presented.

Method of analysis

The polynomial equation of second degree with four unknowns to be solved is

$$xy-2zw+(z-w)^2=(x+y)(z+w) \tag{1}$$

The option

$$x=u+p,\ y=u-p,\ z=u+q,\ w=u-q,\ u\neq \pm p, \pm q \tag{2}$$

in (1) gives

$$p^2+5u^2=6q^2 \tag{3}$$

The procedure to obtain patterns of integer solutions to (1) is as below:

Pattern 1

Assume

$$q=a^2+5b^2 \tag{4}$$

Write the integer 6 as

$$6=(1+i\sqrt{5})(1-i\sqrt{5}) \tag{5}$$

Substitute (4) & (5) in (3). Employing the method of factorization and equating the positive factors, consider

$$p+i\sqrt{5}\,u=(1+i\sqrt{5})(a+i\sqrt{5}\,b)^2$$

Equating the coefficients of corresponding terms, we have

$$p = a^2 - 5b^2 - 10ab$$
$$u = a^2 - 5b^2 + 2ab$$

In view of (2), the integer solutions to (1) are given by

$$\begin{aligned} x &= 2a^2 - 10b^2 - 8ab \\ y &= 12ab \\ z &= 2a^2 + 2ab \\ w &= -10b^2 + 2ab \end{aligned} \qquad (6)$$

A few numerical solutions to (1) are presented in Table 1 below:

Table 1- Numerical solutions.

a	b	x	y	z	w
1	1	-16	12	4	-8
2	3	-130	72	20	-78
3	4	-238	144	42	-136
6	2	-64	144	96	-16
7	5	-432	420	168	-180

Observations

1. $2(x - w) + 25$ is a perfect square
2. Each of the following expressions is expressed as the difference of two squares:

   $\frac{2}{5}(z - x), 2z - y$
3. $12z - 2y$ is a square multiple of 6
4. $12(x - w) + 10y$ is a nasty number
5. The triple $(x, z + w, 4x + 2y)$ is such that the product of any two members added with $\frac{y^2}{4}$ is a perfect square
6. $z + w - x$ is a nasty number when a=2 b
7. $z + w - x = 24t_{3,b}$ when a=b+1
8. $z + w - x = 24P_b^5$ when a=b (b+1)
9. $z + w - x = 72P_b^3$ when a=(b+1) (b+2)
10. $z + w - x = 72P_b^4$ when a=(b+1) (2 b+1)
11. $3x + 2y$ is a square multiple of 6 when $a = 5p^2 + q^2, b = 2pq$

Note

In addition to (5) , the integer 6 may be expressed as

$$6=\frac{(7+i\sqrt{5})(7-i\sqrt{5})}{9}$$

$$6=\frac{(17+i\sqrt{5})(17-i\sqrt{5})}{49}$$

Following the above procedure, two more patterns of integer solutions to (1) are obtained.

Pattern 2

Write (3) as

$$p^2+5u^2=6q^2*1 \quad (7)$$

Assume the integer 1 on the R.H.S. of (7) as

$$1=\frac{(2+i\sqrt{5})(2-i\sqrt{5})}{9} \quad (8)$$

Substitute (4), (5) & (8) in (7). Utilizing factorization and equating positive factors, we have

$$p+i\sqrt{5}\,u=(1+i\sqrt{5})(a+i\sqrt{5}\,b)^2\,\frac{(2+i\sqrt{5})}{3}$$
$$=(-1+i\sqrt{5})(a+i\sqrt{5}\,b)^2$$

On equating the real and imaginary parts in the above equation, one has

$$p=-(a^2-5b^2)-10ab$$
$$u=(a^2-5b^2)-2ab$$

From (2) , the corresponding integer solutions to (1) are given by

$$\begin{aligned} x&=-12ab\\ y&=2a^2-10b^2+8ab\\ z&=2a^2-2ab\\ w&=-10b^2-2ab \end{aligned} \quad (9)$$

Note 2

Apart from (8), the integer 1 may be taken as below:

$$1 = \frac{(5r^2 - s^2 + i\sqrt{5}(2rs))(5r^2 - s^2 - i\sqrt{5}(2rs))}{(5r^2 + s^2)^2}$$

$$1 = \frac{(10s^2 - 10s + 2 + i\sqrt{5}(2s-1))(10s^2 - 10s + 2 - i\sqrt{5}(2s-1))}{(10s^2 - 10s + 3)^2}$$

$$1 = \frac{(2s^2 - 2s - 2 + i\sqrt{5}(2s-1))(2s^2 - 2s - 2 - i\sqrt{5}(2s-1))}{(2s^2 - 2s + 3)^2}$$

$$1 = \frac{(5s^2 - 1 + i\sqrt{5}(2s))(5s^2 - 1 - i\sqrt{5}(2s))}{(5s^2 + 1)^2}$$

The repetition of the above process gives four more sets of integer solutions to (1).

Pattern 3

Consider (3) as

$$6q^2 - p^2 = 5u^2 \tag{10}$$

Assume

$$u = 6a^2 - b^2 \tag{11}$$

The integer 5 on the R.H.S. of (10) is written a

$$5 = (\sqrt{6} + 1)(\sqrt{6} - 1) \tag{12}$$

Substitute (11) & (12) in (10). Applying the method of factorization and equating the positive terms, one has

$$\sqrt{6}q + p = (\sqrt{6} + 1)(\sqrt{6}a + b)^2$$

from which we have

$$q = 6a^2 + b^2 + 2ab$$
$$p = 6a^2 + b^2 + 12ab$$

In view of (2), the integer solutions to (1) are given by

$$\left.\begin{aligned} x &= 12a^2 + 12ab \\ y &= -2b^2 - 12ab \\ z &= 12a^2 + 2ab \\ w &= -2b^2 - 2ab \end{aligned}\right\} \tag{13}$$

Note 3

Apart from (12), the integer 5 may be written as

$$5=(3\sqrt{6}+7)(3\sqrt{6}-7)$$
$$5=(7\sqrt{6}+17)(7\sqrt{6}-17)$$

Repeating the above process, one obtains two more patterns of integer solutions to (1).

Pattern 4

Write (3) as

$$p^2=6q^2-5u^2 \tag{14}$$

Inserting the transformations

$$q=X+5T\ ,u=X+6T \tag{15}$$

in (14) , we have

$$X^2=p^2+30T^2 \tag{16}$$

which is satisfied by

$$T=2rs\ ,p=30r^2-s^2\ ,X=30r^2+s^2 \tag{17}$$

Using (17) in (15), we get

$$q=30r^2+s^2+10rs$$
$$u=30r^2+s^2+12rs$$

In view of (2), the integer solutions to (1) are given by

$$\begin{aligned} x&=60r^2+12rs\\ y&=2s^2+12rs\\ z&=60r^2+2s^2+22rs\\ w&=2rs \end{aligned} \tag{18}$$

Also, express (16) as the system of double equations as in Table 2 below:

Table 2—System of double equations.

System	I	II	III	IV	V	VI	VII	VIII
$X+p$	T^2	$3T^2$	$5T^2$	$15T^2$	$30T$	$15T$	$10T$	$6T$
$X-p$	30	10	6	2	T	$2T$	$3T$	$5T$

Solving each of the above system of equations, the values of X, T, p are obtained. From (15), the values of q and u are determined. From (2), the corresponding integer solutions to (1) are found. For simplicity and brevity, the integer solutions to (1) from solving each of the above system of equations are presented:

Solutions from System I

$x = 4s^2 + 12s, y = 30 + 12s, z = 4s^2 + 30 + 22s, w = 2s$

Solutions from System II

$x = 12s^2 + 12s, y = 10 + 12s, z = 12s^2 + 10 + 22s, w = 2s$

Solutions from System III

$x = 20s^2 + 12s, y = 6 + 12s, z = 20s^2 + 6 + 22s, w = 2s$

Solutions from System IV

$x = 60s^2 + 12s, y = 2 + 12s, z = 60s^2 + 2 + 22s, w = 2s$

Solutions from System V

$x = 72s, y = 14s, z = 84s, w = 2s$

Solutions from System VI

$x = 42s, y = 16s, z = 56s, w = 2s$

Solutions from System VII

$x = 32s, y = 18s, z = 48s, w = 2s$

Solutions from System VII

$x = 24s, y = 22s, z = 44s, w = 2s$

CONCLUSION

Plenty of solutions in integers are presented in this paper for the quaternary homogeneous quadratic equation given by $xy - 2zw + (z - w)^2 = (x + y)(z + w)$ through employing substitution technique and factorization method. As quadratic equations (homogeneous or non-homogeneous) are plenty, one may search for patterns of integer solutions to other choices of multivariable quadratic equations.

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